Vladimir M. Ristić

Noether's Theorem

A physicists view at and
new gnosiological aspects of the famous
theorem

ISBN 978-86-81829-93-6

Editorial supervision
Nataša Nedeljković
Ivan Mančev

Published by Faculty of Science, Kragujevac
Faculty of Science Kragujevac Series in Science
Editor
Dragoslav Nikezić, Dean

Kragujevac, 2008

To the loving memory of my late wife Nena,

Who has bestowed me with two wonderful children

Cassandra and Mathew

CONTENTS

Preface

Jorge Luis Borges has pointed out both, explicitly, writting about his poetics and, implicitly, in his short stories, that in writing a book, which was considered before him as exclusively an individual act, there is a collective contribution. This is also true for scientific research, and there is no better example of that than Noether's theorem, which, though an individual invention, is now included in the collective heritage of mankind, not only as an important mathematical theorem, but as a living organism which pulsates and iradiates ever new and new results.

Noether's theorem is that astonishing example of "penetrating thinking" [1], which is equaly suited to both phylosophical, especially gnosiological, analysys and deep group mathematical analysys. So it could be used in various fields of physics, because it is open to hermeneutical applications to new theory types (ch. 5), to new interpretations of already accepted results (ch. 6), and maybe before all, to understanding the roots of our civilisation and connecting the two principles of causality: Greek and Western (see, ch.4) by resolving the problem of the connservation of the energy in the general relativity theory. The symmetry group of the general relativity theory is a Lie group with a continuously infinite number of independent infinitesimal generators (Emmy Noether in her paper [see,2] calls it infinite continuus group), while the symmetry group of special relativity is the Poincare group, a Lie subgroup of the group of general coordinate transformations, which has a finite number (7) of independent infinitesimal generators (finite continuous group in Noether's paper). This is what makes the difference between Noether's theorem I and II: theorem I describes the case when there is a finite continuous group of symmetries, and therem II deals with cases covering an infinite continuous group of symmetries. And yet another feature of contemporary group theory, namely active and passive interpretation [22], gives possibility to include both priciples of causality: Greek and Western, into the interpretation of both theories of relativity via Noether's theorem (see ch. 4). Maybe some clarifying of the earlier statments is needed, so we shall shortly discuss the problem of two variants of the principle of causality.

Instead of general formulation “equal causes have equal effects” the Western principle of causality might be formulated as "Every **change** of the *state* of the *object* has its cause", while the Greek principle states "Every **object** has its cause”[1] (both formulations were taken over from private communications with Professor V. Bočvarski, and so are many definitions of Greek views in ch.4). To illustrate this one could talk about the process of obtaining water steam. The Western explanation of the process is, to put it simply, that water, by some definite cause (heating, pressure etc.) is going over into steam, but for Greek thought water **disappears**

[1] In fact according to Aristotle it can have four causes: *the efficient, formal, material and final*, but each of them is connected with the “object” itself, and not with the “change of the state of object”. Galileo-Newtonian physics has kept only the efficient cause but connecting it to the “change of the state of object”.

while steam **appears**. The cause is also heating, though they were not aware of the influence of pressure, so the two principles do not seem to be so different. Yet, the Bohr Postulates (Published in 1913) include into themselves the notions of ***appearing*** and ***disappearing*** of objects (without mentioning the connection with Greek principle of causality, and probably without ever noticing it). Electrons in their orbits, when changing the orbit are behaving themselves like "Greek objects", i.e. *disappearing* and *appearing*, without any reference to the period of time (here should be stressed that the point of time was made by later thinkers -Western- and was not raised by Greeks, who speaking of time had in mind *existence*) and portion of space in which they should be, according to Galileo-Newtonian approach, during this "change". And this is not only the case in the Old Quantum Theory, but also in the Matrix Mechanics of Heisenberg. Namely, when Heisenberg decided to incorporate definitely the correspondence principle into his new theory, so that it would not be needed to reach for in solving each new quantum problem, which was done in the case of Old Quantum Theory, and, especially, when he obtained the orbits of the electrons, once for all, as the eigenvalues of matrices that represent physical quantities, he established the foundation for laying down the Greek principle of causality underneath all formulations of Quantum Mechanics. Accordingly in Schrödinger's Wave Mechanics, which was shown, even in the first papers of Schrödinger, to be equivalent to the Matrix Mechanics (and shown to be the one of the possible representations of the abstract Quantum Theory realized via the Hilbert space, by Dirac, and proven finally and conclusively to be so by von Neumann), the Greek principle of causality was accepted through the eigenvalues of operators (which were introduced into Quantum theory some time before that by Born and Wigner, for all this and other historical facts one should see [3]), representing physical quantities. Eigenvalues of those operators represent the orbits of electrons that are changed by *disappearing* and *appearing* of electrons. And there comes in the "Copenhagen interpretation" which pushes further Heisenberg's try via the *Uncertainty Relations* to connect this two different causality principles (unfortunately, without explicitly noticing it). The Copenhagen interpretation of Quantum Mechanics tries to make this connection by introducing *probability*, thus avoiding making choice between two concepts. This is, probably, why the "Copenhagen interpretation" was so unacceptable for Einstein intuitively (he, of course, tried to realize his intuition introducing EPR-paradox, etc.). But, as is already known, Einstein's tries failed in a sense that Quantum Mechanics survived. And this survival was so overwhelming that we now have in Relativistic Quantum Mechanics, or in Quantum Field Theory, lots of operators of creation and annihilation, charge operators which create and annihilate particles and so on.

But this relatively clear situation is complicated when observing the theory of relativity from the point of view of the two definitions of causality, Greek and Western. Because, relativity introduces the space and time which are tied to the referent system, i.e. carried with objects themselves. So it seems that the Greeks, who never considered space in a modern sense, but only some kind of place carried with the object itself, and, as mentioned earlier never introduced time into treatment of motion[2], have succeeded in some mysterious way in sneaking their notion of causality into yet another modern physical theory. Though, until now understanding of it could be based only on intuition, which is not good enough for any, even slight, scientific approach. Nevertheless, looking into Noether's theorem from another point of view could give us ground for firmer foundation of such views, see ch.4.

[2] That is, never in the Eleatic tradition, which was implicitly included into Plato's and Aristotle's teachings. Aristotle himself had, in his *Physics*, defined something like velocity, but never really managed to resolve Zeno's paradoxes using his own concept of time – based on existance. Today mathematicians claim that they can resolve Zeno's paradoxes using modern calculus, which through the notion of function includes into itself the modern concept of time [see 4.3.].

1. Introduction

She should have died hereafter;
There would have been time for such a word.
Shakespeare, *Macbeth*, Act 5, Scene 5

Amalie Emmy Noether was born at the end of XIX century in the town of Erlangen in that tame German country, which was praised in such manner by Heinrich Böll, one of German's most prominent writers of the end of XX century.

Exactly speaking she was born on March 23, 1882 in Erlangen, Bavaria, Germany, and died at the age of 53 on April 14, 1935 in Bryn Mawr, Pennsylvania, USA. Short explanation of these biographical data follows.

It is not a rare thing to meet the author in mathematics, and related fields so farsighted as was Emmy Noether. Nor so deep in her apprehansion. But Emmy Noether, when she in 1918 submitted to the German Scientific Society paper entitled "Invarianten Varlationsprobleme" [2], was a young bright woman who lived in a society not open to women's emancipation. In Wilhelm's Germany the proverb "Kinder, Küche und Kürche", meaning "Children, kitchin and church", was overwhelmingly popular in explaining women their place in the society.

So she started her education attending the Höhere Töchter Schule (High Teachers School) in Erlangen from 1889 to 1897. She studied German, English, French, arithmetics and took piano lessons. She conntinued to study English and French and she, when became of age of around 18, was a certified teacher of English and French for Bavarian girl schools. But, even so, she never actually taught languages. Instead she took the difficult way of studying mathematics at university, difficult in the light of women's position in Germany at that time as mentiond before. Women were allowed to study at German universities only unofficially (for instance, each professor had to give permission for his course). Emmy Noether obtained permission to sit on courses at the University of Erlangen during 1900 to 1902, after which, having taken and passed the matriculation examination at Nürnberg in 1903, she began to attend the University of Götingen, visiting lectures of Blumenthal, Hilbert, Klein and Minkowski. Later in 1904 she was permitted to matriculate at Erlangen, and was working under Paul Gordan on her doctorate which was granted to her in 1907. Gordan was working, same as Hilbert, on existance of finitness of invariants in n varaiables, but, unlike Hilbert, took a constructive approach, which was followed by Emmy Noether in her doctoral thesis, where she listed systems of 331 covariant forms.

But, for the reasons mentioned earlier, she could not follow the normal progression to an academic post. Namely, being a woman, she could not get habilitation. Instead she remained at Erlangen, helping her father, which was not so well, and was very pleased with his dauther's help, though this was not the general feeling of Emmy herself. Even so she kept her spirits up and worked on her own research, now influenced by Fischer who had succeded Gordan in Erlangen in 1911, and who inclined towards Hilbert's abstract approach, persuading thus Emmy Noether away from the constructive approach of Gordan.

Then she began to publish her own results and her reputation got a considerable boost very quickly. Circolo Mathematico di Palermo elected her for its membership in 1908, and after that in 1909 she was invited to become a member of the Deutche Mathematiker-Vereinigung, and immediately, the same year she adressed the annual meeting of the Society in Salzburg. Consecvently in 1913 she lectured in Vienna.

Yet all this was far from getting habilitation at Göttingen, so Emmy Noether was excited when in 1915 Hilbert and Klein invited her to return to Göttingen. Persuaded by them there she remained while they fought a long and exauhsting battle to have her officially on the Faculty. Using the word "battle" is not over exaggerated here, because fighting with university authorities to allow Emmy Noether to obtain her habilitation lasted until 1919, and there where many setbacks and slow progressing in achieving permission to be granted. But Emmy Noether was encouraged during this time with being allowed by Hilbert to lecture advertising her courses under his own name. For instance a course given in the winter semester of 1916-17 is catalogiesed as:

Mathematical Physics Seminar: Professor Hilbert, with the assistance of Dr. E. Noether.

Mondays from 4-6, no tuition.

Somewhere inbetween, she has presented at the July 16, 1918 meeting of the Königsche Gesellschaft der Wissenschaften zu Götingen her paper *Invariante Varlationsprobleme* [2] where she has proved two famous theorems, and their converses, on the connection between symmetries and conservation laws. In fact, the paper was presumably presented by Felix Klein, because Emmy Noether was not the member of that Society. Theorems where not easy to grasp but had an undeniable influence on the contemporary physics, introducing the more profound understanding of conservation of energy, momentum, angular momentum and so on. At the time the most important result of these theorems was, as Emmy Noether stated it, a bit underestimating her result [quoted after 2]:"From the foregoing ... [we] obtain the proof of an assertion of Hilbert concerning the connection between the failure of proper energy conservation laws and general relativity, and indeed in a general group – theoretic setting." So the Noether theorem began its life as a metatheorem, proving that the general theory of relativity is obeying the law of conservation of energy, not violating it as generaly was douted.

In contemporary terminology, the symmetry group of the general relativity theory is a gauge group of all continous coordinate transformations with continuous derivatives (a group of general coordinate transformations). So, it is a Lie group with a continuously infinite number of independent infinitesimal generators (Emmy Noether in her paper calls it infinite continuus group), while the symmetry group of special relativity is the Poincare group, a Lie subgroup of the group of general coordinate transformations, which has a finite number (7) of independent infinitesimal generators (finite continuous group in Noether's paper). This is what makes the difference between Noether's theorem I and II: theorem I describes the case when there is a finite continuous group of symmetries, and therem II deals with cases covering an infinite continuous group of symmetries. Hilbert called "proper energy theorems" the conservation laws obtained from field theories that could be described by Noether's I theorem, because, physically in such theories the energy density which is conserved is localized, as the energy-momentum tensor of such theories is divergence free. Yet it is meaningless to speak in *general relativity* of definite localization of energy, because the tensor analogous to the energy-momentum density tensor of *special relativity*, can be made divergence free, but it is gauge dependent, thus not being covariant under general coordinate transformations, or in Hilbert's words in such theories one has "improper energy theorems". These are the points that Emmy Noether clarified in her historical paper of 1918 [2], or as Feza Gursey wrote [quoted according to 2]: "Before Noether's theorem the principle of conservation of energy was shrouded in mystery, leading to the obscure physical systems of Mach and Ostwald. Noether's simple and profound mathematical formulation did much to demistify physics."

At Göttingen, after obtaining habilitation in 1919, Noether left the invariant theory and started to work on ideal theory. This her work helped develop ring theory into major mathematical topic. Her study of chain conditions on the ideals of rings, also of the Associative Law, the Commutative Law and the Distributive Law made them the powerful tools of mathematical research. Her paper *Idealtheorie in Ringbereichen* (1921) fundamentally inflenced the development of modern algebra.

In 1924 B.L. van der Waerden came to Göttingen. Emmy Noether and the famous Dutch mathematician spent a year studying togather. As a result, van der Waerden wrote his book *Modern Algebra* in two volumes, of which the second one mostly consists of Noether's work. It is caractheristic of Emmy Noether that much of her work is first published in papers by colleagues and students, avoiding her own name.

Nevertheless, there was awareness in the mathematical community at the time of her enormous mathematical achivements, and it resulted in invitations to adress the International Mathematical Congress at Bologna in 1928; than again at Zürich in 1932. Also in 1932 she, jointly with Artin, received the Alfred Ackermann-Teubner Memorial Prize for the Advancement of Mathematical Knowledge.

But even so, the fate of Ammaly Noether, one of the most successful woman mathematicians, was not developing further as it was expected. Though successful in convincing University authorities to accept her as a valuable mathematicion, she, in year 1933, even never tried to convince the Nazi authorities that she was not Jewish, and had been dismissed from the University of Göttingen. As many Jewish scientists at that time she turned to USA, and accepted a visiting professorship at Bryn Mawr College, also lecturing at the Institute for Advanced Study, Princeton.

There, once again she organized her life, teaching and working on the problems she brought with her. The students always followed her around. They were known, as Noether's flock.

2. Easy approach

…there is a special providence in the fall of a sparrow.
Shakespeare, *Hamlet*, Act 5, Scene 2

As mentioned earlier both Emmy Noether's theorems are demanding rather complicated mathematics, but are stunningly simple in their outcome. To adjust smoothly that two unpleasantly opposed sides of this elegant peace of mathematical thinking, physicists, as they usually do with mathematics that is needed for their research, have tried to simplify the problem by merging the two theorems proved by Noether in 1915, and presented publicly only in 1918, into one usually called 'Noether's theorem'. Besides that, physicists, who often like to simplify the mathematics they use beyond its and theirs possibilities, have noticed the following:

If using the elegant methods of Lagrangian Mechanics [4], one can write down the equation of motion for a simple system described with Lagrangian L of the variables q and $\dot{q}$ = dq/dt as expression

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)=\frac{\partial L}{\partial q}, \qquad (2.1)$$

which is widely known as the Euler-Lagrange equation of motion, and as being equivalent to the statement that action of the system is equal to zero, according to Hamilton's principle. But yet, here, in the word 'equivalent' there is a trap which has directed many physicists to the following not fully justified simplification of Noether's theorem (one of the many examples could be found in [5]). Namely, the most frequent statement which could be found in such approaches is that equation (2.1) signifies the following: if the quantity on the right hand side is zero, the quantity in the parentheses on the left hand side is a conserved quantity. This proposition is then called Noether's theorem. And formally speaking one can thus obtain many results that are produced by Noether's theorem. For instance, having in mind that the classic Lagrangian of a free particle is just $L = \frac{1}{2} m\dot{x}^2$, and depends only on $\dot{x}$, not on x , so dL/dx = 0, it follows from Eq. (1) that $dL/d\dot{x} = m\dot{x}$ is constant, i.e. the x projection of momentum is conserved. So can be proved for other components, and it can be easily extended to a system of particles with n degrees of freedom. I.e. the conservation of momentum in general follows from (1). For the similar set of particles the invariance of Lagrangian under a fixed rotation of the system about the origin, could also be considered. In such case if coordinates of a point mass were x,y,z a fixed rotation about the z axis preserves the quantity $x^2 + y^2$ for each point. Differentiating the last expression one gets xdx + ydy = 0, so, introducing differentials, in accordance to the symmetry involved, by $dx_i = d\sigma y_i, dy_i = -d\sigma x_i$, and $dz_i = 0$, by eq. (1) one has a conserved quantity

$$\sum_{i=1}^{N} m_i(\dot{x}_i y_i - x_i \dot{y}_i) = \text{constant}, \qquad (2.2)$$

which is nothing else but the conservation of angular momentum about the z axis. In the similar manner one can show the conservation of angular momentum about the x or y axes.

But the angular momentum and spin conservation could not be shown in that manner, because for that it is needed to take into account rotation of infinitesimal 4-element. This shows the limitations of the simplified approach to Noether's theorem. Of course the simplification is contented in the expression "equivalence" when referring to equation (2.1) and its relation to Hamilton's principle, which is the cause of that equation, and thus, even by simple logic both cannot be fully equivalent.

3. Heavy approach

And enterprises of great pith and moment…
Shakespeare, *Hamlet*, Act 3, Scene 1

Now, following [6], the Noether's theorem (the first one) in its full capacity will be proved (I do thank my assistant Msc. Mirko Radulović, for helping me with this proof). The formulation of the theorem, most suitable to physicists (though not minimal in its statement) is:

To any S-parametric continuous transformation of field functions and coordinates, which keeps variation of action zero, and for which the law of transformation of field functions is known, there correspond S-dynamic invariants (i.e. combinations of field functions and their derivatives), which remain constant in time.

Either, as the original statement of the theorem is "not so easy to be found", the minimal statement could be:

To every symmetry group transformation, there corresponds a conserved current. [7]

Or [8]

If system (M, L) has one parameter group of diffeomorphisms $h^s : M \to M, s \in \Re, h^0 = E$, *the corresponding system* of L *Lagrangian equations has the first integral* I: $TM \to \Re$.

We shall go back to the first of our definitions [6], which is, to our opinion, most suitable for physicists. To prove it let us start [following 6] with an infinitesimal transformation of coordinates

$$\bar{x}^{\kappa} \to x^{\kappa} + \delta x^{\kappa},\ \delta x^{\kappa} = \sum_{1\leq j\leq s} X_j^{\kappa} \delta x^{j} \,. \qquad (3.1)$$

Here and throughout this book we shall use Greek indices to denote 4 space-time coordinates (0,1,2,3) and Latin for spatial 3-coordinates (1,2,3), and also, when indicated, Latin indices will be used for numerating much greater values, as in the sum on the left hand side of equation (3.1).

§3.1. PROVING I NOETHER'S THEOREM

The law of transformation of field functions, for (3.1), is given by

$$u_{\alpha}(x) \to \bar{u}_{\alpha}(\bar{x}) = u_{\alpha}(x) + \delta u_{\alpha}(x) \,.$$

Here the variation of field functions, caused by the change of the shape of function, as well as by the change of the argument, is

$$\delta u_{\alpha}(x) = \sum_{1\leq j\leq s} \psi_{\alpha j} \delta \omega^{j} \,, \qquad (3.2)$$

where X_j^{κ} in (3.1) and $\psi_{\alpha j}$ are transformation matrices. Using Einstein's summation convention, which shall be used throughout this book, one can write down the variation due to the shape of the function

$$\delta u'_\alpha \equiv \bar{u}_\alpha(x) - u_\alpha(x) = \left(\psi_{\alpha\nu} - \frac{\partial u_\alpha}{\partial x^\kappa} X^\kappa_\nu \right) \delta\omega^\nu .$$

Then variation of Action is

$$\delta A = \delta \int L(x) dx .$$

As variation and integration are mutually independent operations, it follows

$$\delta A = \int \delta(L(x) dx) = \int \delta L(x) dx + L(x) \delta(dx) . \quad (3.3)$$

The first term in eq. (3.3) is due to the variation of Lagrangian, the second due to the variation of area of integration.

(a) Variation of area of integration gives the following expression

$$\delta(dx) = \delta\left(\frac{\partial x}{\partial x^\lambda} dx^\lambda \right) = \delta\left(\frac{\partial x}{\partial x^\lambda} \right) dx^\lambda + \frac{\partial x}{\partial x^\lambda} \delta(dx^\lambda) .$$

On the other hand, as operations of varying and differentiating are mutually independent, it follows, with the help of some dummy indices gymnastics,

$$\delta(dx) = \frac{\partial x}{\partial x^\lambda} d(\delta x^\lambda) = \frac{\partial x}{\partial x^\lambda} \frac{\partial(\delta x^\lambda)}{\partial x^\kappa} dx^\kappa = \frac{\partial x}{\partial x^\lambda} dx^\lambda \frac{\partial(\delta x^\lambda)}{\partial x^\lambda}$$

or, finally,

$$\delta(dx) = dx \frac{\partial(\delta x^\lambda)}{\partial x^\lambda} . \quad (3.4)$$

(b) Variation of the Lagrangian could be obtained in general as

$$\delta L = \bar{L}(\bar{x}) - L(x) = \bar{\delta} L + \frac{\partial L}{\partial x^{\beta}} \delta x^{\beta}.$$

Here $\bar{\delta} L$, the variation of form of the Lagrangian, is given by

$$\bar{\delta} L = \bar{L}(x) - L(x) = \frac{\partial L}{\partial u_{\alpha}} \bar{\delta} u_{\alpha} + \frac{\partial L}{\partial \left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}} \right)} \bar{\delta} \left(\frac{\partial u_{\alpha}}{\partial x^{k}} \right).$$

From Euler-Lagrange equation (2.1) in generalized form:

$$\frac{\delta A}{\delta u_{\alpha}} = \frac{\partial L}{\partial u_{\alpha}(x)} - \frac{\partial}{\partial x_{k}} \left(\frac{\partial L}{\partial \left(\frac{\partial u_{i}}{\partial x^{\kappa}} \right)} \right) = 0 ,$$

it follows

$$\frac{\partial L}{\partial u_{\alpha}} = \frac{\partial}{\partial x_{\kappa}} \left(\frac{\partial L}{\frac{\partial u_{i}}{\partial x^{k}}} \right),$$

so one has, interchanging variation and differentiation,

$$\bar{\delta} L = \frac{\partial}{\partial x^{\kappa}} \left(\frac{\partial L}{\partial \left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}} \right)} \right) \bar{\delta} u_{\alpha} + \frac{\partial L}{\partial \left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}} \right)} \frac{\partial}{\partial x^{k}} (\bar{\delta} u_{\alpha}) .$$

Thus the variation of the form of the Lagrangian can be written down in the following manner

$$\bar{\delta} L = \frac{\partial}{\partial x_\kappa}\left(\frac{\partial L}{\partial\left(\frac{\partial u_\alpha}{\partial x^k}\right)} \bar{\delta} u_\alpha \right). \tag{3.5}$$

Using expressions (3.4) and (3.5), one can show that the variation of Action is given by

$$\delta A = \int \left\{ \frac{\partial}{\partial x^\kappa}\left(\frac{\partial L}{\partial\left(\frac{\partial u_\alpha}{\partial x^\kappa}\right)} \bar{\delta} u_\alpha \right) + L(x)\frac{\partial (x^k)}{\partial x^\kappa} \right\} dx =$$

$$= \int \left\{ \frac{\partial}{\partial x^\kappa}\left(\frac{\partial L}{\partial\left(\frac{\partial u_\alpha}{\partial x^\kappa}\right)} \bar{\delta} u_\alpha \right) + \frac{\partial}{\partial x^\kappa}[L(x)\delta x^\kappa] \right\} dx =$$

$$= \int \frac{\partial}{\partial x^{\kappa}} \left[\frac{\partial L}{\partial \left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}} \right)} \bar{\delta} u_{\alpha} + [L(x) \delta x^{\kappa}] \right] dx \,.$$

As variations in the above expression could be expressed as

$$\bar{\delta} u_{\alpha}(x) = \sum_{j} \left(\psi_{\alpha j} - \frac{\partial u_{\alpha}(x)}{\partial x^{\kappa}} X_{j}^{\kappa} \right) \delta \omega^{j}; \; \delta x^{\kappa} = \sum_{j} X_{j}^{\kappa} \delta \omega^{j},$$

our expression for action becomes

$$\delta A = \int \frac{\partial}{\partial x^{\kappa}} \left[\frac{\partial L}{\partial \left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}} \right)} \sum_{j} \left(\psi_{\alpha j} - \frac{\partial u_{\alpha}(x)}{\partial x^{\kappa}} X_{j}^{\kappa} \right) \delta \omega^{j} + L(x) \sum_{j} X_{j}^{\kappa} \delta \omega^{j} \right] dx =$$

$$= \int \sum_{j} \frac{\partial}{\partial x^{\kappa}} \left[\frac{\partial L}{\partial \left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}} \right)} \left(\psi_{\alpha j} - \frac{\partial u_{\alpha}(x)}{\partial x^{\kappa}} X_{j}^{\kappa} \right) + L(x) X_{j}^{\kappa} \right] \delta \omega^{j} dx =$$

$$= - \int \sum_{j} \frac{\partial}{\partial x^{\kappa}} \left[\frac{\partial L}{\partial \left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}} \right)} \left(\psi_{\alpha j} - \frac{\partial u_{\alpha}(x)}{\partial x^{\kappa}} X_{j}^{\kappa} \right) - L(x) X_{j}^{\kappa} \right] \delta \omega^{j} dx =$$

$$= - \int \sum_{j} \frac{\partial \Theta_{j}^{\kappa}}{\partial x^{\kappa}} dx \, \delta \omega^{j}, \qquad (3.6)$$

where

$$\Theta^{\kappa}_{(\beta)}(x) = -\frac{\partial L}{\partial\left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}}\right)}\left(\frac{\partial u_{\alpha}(x)}{\partial x^{\kappa}}X^{\kappa}_{j} - \psi_{\alpha j}\right) - L(x)X^{\kappa}_{j}$$

(3.7)

(β)- represents an index which can take on, besides standard 4 values, value equal to the number of continuous parametric transformations, s.

Now, as stated in Noether's theorem, we demand that variation of action vanishes, i.e. that action has an extremal (minimal) value

$$\frac{\delta A}{\delta\omega^{\beta}} = -\int \frac{\partial\theta^{\kappa}_{(\beta)}}{\partial x^{\kappa}}\,dx = 0,$$

implicating,

$$-\frac{\partial\theta^{\kappa}_{(\beta)}(x)}{\partial x^{\kappa}} = 0 \qquad (3.8)$$

It is possible from the above expression, using Gauss' theorem, to obtain the conservation of corresponding space integrals. If (3.8) is integrated over volume (infinite in its spacelike part, but limited with two threedimensional surfaces σ_1 and σ_2, in its timelike part), assuming that at the limits of spacelike volume the field is being zero, it follows

$$\int_{\sigma_1} \Theta^{\kappa}_{(\beta)} \cdot d\sigma_{\kappa} = \int_{\sigma_2} \Theta^{\kappa}_{(\beta)} \cdot d\sigma_{\kappa} . \tag{3.9}$$

Here $d\sigma_{\kappa}$ is a projection of the element of surface σ on a three-surface, normal to the direction of x^{κ}-axis. Equation (3.9) shows that surface integrals of the form

$$C_{(\beta)}(\sigma) = \int_{\sigma} \Theta^{\kappa}_{(\beta)} d\sigma_{\kappa} ,$$

are not depending on the shape of the surface σ. In the case when surface σ is a three-surface $x^0 \equiv t = \text{const}$, integration is carried over three-dimensional configuration space, and integrals

$$C_{(\beta)}(x^0) = \int_{\sigma} \Theta^{0}_{(\beta)} d\mathbf{x} ,$$

are time independent.

Thus, it has been shown that to every s-parametric continuous transformation of coordinates and field functions corresponds a certain time independent invariant $C_i (i = 1, \cdots, s)$, i.e. the I Noether's theorem has been proved.

Maybe here it is convinient to stress that there is possibility of proving, so to say, Noether's theorem Ia i.e. the case of countably infinite number of parameters, as Noether's theorem II is connected with continuously infinite number of parameters. As the countably infinite case is very similar to the one of finite number of parameters, it is obvious that this inbetween theorem is easy to proove in the manner very similar to the one above.

§3.2. 4-MOMENTUM TENSOR

Using the example of infinitesimal quadri-rotations we shall illustrate the preceding discussion of I Noether's theorem. In the case of infinitesimal quadri-rotations the change of coordinate system is given by

$$\bar{x}^{\kappa} = x^{\kappa} + x_{\lambda}\delta\omega^{\kappa\lambda}. \qquad (3.10)$$

As parameter's ω are antisymmetric, i. e.

$$\delta\omega^{\nu\lambda} = -\delta\omega^{\lambda\nu},$$

those six that are linearly independent will only be used: $\delta\omega^{\nu\lambda}$,($\nu<\lambda$).Indices ν and λ shall mark the surface of rotation, and rotation parameters shall be $\omega^{\nu\lambda}$. So, index (j) in (3.1) and (3.2) splits into two indices:

$$(j) \longrightarrow \nu, \lambda,$$

and eq. (3.1) becomes

$$\delta x^{\kappa} = X^{\kappa}_{(\nu\lambda)} \delta\omega^{(\nu\lambda)} \qquad (3.11)$$

On the other side, from expression (3.10) it follows

$$\delta x^{\kappa} = x_{\lambda}\delta\omega^{\kappa\lambda} = x_{\lambda}\delta\omega^{\mu\lambda}\delta^{\kappa}_{\mu} = \sum_{\mu<\lambda} x_{\lambda}\delta\omega^{\mu\lambda}\delta^{\kappa}_{\mu} + \sum_{\mu>\lambda} x_{\lambda}\delta\omega^{\mu\lambda}\delta^{\kappa}_{\mu} =$$
$$= \sum_{\mu<\lambda} x_{\lambda}\delta\omega^{\mu\lambda}\delta^{\kappa}_{\mu} + \sum_{\mu>\lambda} x_{\mu}\delta\omega^{\lambda\mu}\delta^{\kappa}_{\lambda},$$

and, taking into account antisymmetry of $\omega^{\mu\lambda}$, we get

$$\delta x^{\kappa} = \sum_{\mu<\lambda} x_{\lambda}\delta\omega^{\mu\lambda}\delta^{\kappa}_{\mu} - \sum_{\mu<\lambda} x_{\mu}\delta\omega^{\mu\lambda}\delta^{\kappa}_{\lambda} = \sum_{\mu<\lambda}(x_{\lambda}\delta^{\kappa}_{\mu} - x_{\mu}\delta^{\kappa}_{\lambda})\delta\omega^{\mu\lambda}$$
(3.12)

From above equation it follows that the next expression can be written down

$$X^{\kappa}_{\mu\lambda} = x_{\lambda}\delta^{\kappa}_{\mu} - x_{\mu}\delta^{\kappa}_{\lambda} \quad (\mu \le \lambda).$$

Next we shall discuss the change of field functions for infinitesimal quadri-rotations. Analogously to eq. (3.10) one has

$$\bar{u}_{\alpha}(\bar{x}) = u_{\alpha}(x) + \delta u_{\alpha} \quad \delta u_{\alpha} = \sum_{\alpha,\kappa<\lambda} A^{\nu}_{\alpha\kappa\lambda} u_{\nu}(x)\delta\omega^{\kappa\lambda}.$$

so it is clear that the variation of field functions is given as

$$\delta u_\alpha(x) = \sum_{\nu,\kappa<\lambda} A^{\nu}_{\alpha(\kappa\lambda)} u_\nu(x) \delta\omega^{\kappa\lambda} . \qquad (3.13)$$

For scalar field $A^{\nu}_{\alpha(\kappa\lambda)} = 0$, while for the covariant vector field

$$A^{\nu}_{\alpha(\kappa\lambda)} = g_{\alpha\kappa}\delta^{\nu}_{\lambda} - g_{\alpha\lambda}\delta^{\nu}_{\kappa}, (\kappa < \lambda) .$$

From expression (3.2) one knows that

$$\delta u_\alpha(x) = \Psi_{\alpha(\nu\lambda)} \delta\omega^{\nu\lambda} .$$

Comparing the above expression with equation (3.13), one has

$$\Psi_{\alpha(\kappa\lambda)} = A^{\nu}_{\alpha(\kappa\lambda)} u_\nu(x) = g_{\alpha\kappa} u_\nu(x) \delta^{\nu}_{\lambda} - g_{\alpha\lambda} u_\nu(x) \delta^{\nu}_{\kappa} .$$

So, finally, one has

$$\Psi_{\alpha(\kappa\lambda)} = g_{\alpha\kappa} u_\lambda - g_{\alpha\lambda} u_\kappa .$$

From equation (3.7), with little index gymnastics, could be obtained

$$M^{\kappa}_{(\nu\lambda)} = \frac{\partial L}{\partial\left(\frac{\partial u_\alpha}{\partial x^\kappa}\right)}\left(\frac{\partial u_\alpha}{\partial x^\beta} X^{\beta}_{(\nu\lambda)} - \psi_{\alpha(\nu\lambda)}\right) - L X^{\kappa}_{(\nu\lambda\}}.$$

Introducing expressions for $X^{\kappa}_{(\nu\lambda)}$ and $\Psi_{\alpha(\nu\lambda)}$ into above equation

$$M^{\kappa}_{(\nu\lambda)} = \frac{\partial L}{\partial\left(\frac{\partial u_\alpha}{\partial x^\kappa}\right)}\left(\frac{\partial u_\alpha}{\partial x^\beta}(g^{\lambda\lambda} x^{\lambda} \delta^{\beta}_{\nu} - g^{\nu\nu} x^{\nu} \delta^{\beta}_{\lambda}) - A^{\gamma}_{\alpha(\nu\lambda)} u_\gamma\right) -$$
$$- L(g^{\lambda\lambda} x^{\lambda} \delta^{\beta}_{\nu} - g^{\nu\nu} x^{\nu} \delta^{\beta}_{\lambda}),$$

and performong minor transformations, one obtains

$$M^{\kappa}_{(\nu\lambda)} = \frac{\partial L}{\partial\left(\frac{\partial u_\alpha}{\partial x^\kappa}\right)} g^{\lambda\lambda} x^{\lambda} \frac{\partial u_\alpha}{\partial x^\nu} - \frac{\partial L}{\partial\left(\frac{\partial u_\alpha}{\partial x^\kappa}\right)} g^{\nu\nu} x^{\nu} \frac{\partial u_\alpha}{\partial x^\lambda} - \frac{\partial L}{\partial\left(\frac{\partial u_\alpha}{\partial x^\kappa}\right)} A^{\gamma}_{\alpha(\nu\lambda)} u_\gamma -$$
$$- L(g^{\lambda\lambda} x^{\lambda} \delta^{\beta}_{\nu} - g^{\nu\nu} x^{\nu} \delta^{\beta}_{\lambda}),$$

and, finally,

$$M^{\kappa}_{(\nu\lambda)} = g^{\lambda\lambda} x^{\lambda} \left(\frac{\partial L}{\partial\left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}}\right)} \frac{\partial u_{\alpha}}{\partial x^{\beta}} - L\delta^{\beta}_{\nu} \right) - g^{\nu\nu} x^{\nu} \left(\frac{\partial L}{\partial\left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}}\right)} \frac{\partial u_{\alpha}}{\partial x^{\beta}} - L\delta^{\beta}_{\lambda} \right)$$

$$- \frac{\partial L}{\partial\left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}}\right)} A^{\gamma}_{\alpha(\nu\lambda)} u_{\gamma}.$$

Denoting expressions in parenthesis as T^{κ}_{ν} and T^{κ}_{λ} , respectively, one finally obtains the explicit form of 4-momentum tensor

$$M^{\kappa}_{(\nu\lambda)} = g^{\lambda\lambda} x^{\lambda} T^{\kappa}_{\nu} - g^{\nu\nu} x^{\nu} T^{\kappa}_{\lambda} - \frac{\partial L}{\partial\left(\frac{\partial u_{\alpha}}{\partial x^{\kappa}}\right)} A^{\gamma}_{\alpha(\nu\lambda)} u_{\gamma} . \quad (3.14)$$

From equation (3.14) it is easy to see connection between symmetry of energy-momentum tensor and the structure of 4-angular momentum tensor:

(a) For scalar field, the third term in (3.14) is equal to zero, so one has

$$M^{k}_{(\nu\lambda)} = g^{\lambda\lambda} x^{\lambda} T^{\kappa}_{\nu} - g^{\nu\nu} x^{\nu} T^{\kappa}_{\lambda}$$

and, finally

$$M^{\kappa}_{(\nu\lambda)} = x^{\lambda} T^{\nu\kappa} - x^{\nu} T^{\lambda\kappa}, \qquad (3.15)$$

which is internal, *angular momentum* of the wave field.

(b) For multicomponent fields (vector, spinor) expression for $M^{\kappa}_{(\nu\lambda)}$ is not changing and its third term is describing the polarization qualities of the field, i.e. it represents the spin momentum of the particle describing quantized field:

$$S^{\kappa}_{\nu\lambda} = -\frac{\partial L}{\partial\left(\dfrac{\partial u_{\alpha}}{\partial x^{\kappa}}\right)} A^{\gamma}_{\alpha(\nu\lambda)} u_{\gamma} . \qquad (3.16)$$

Thus one can see the rich structure underlying Noether’s theorem is giving the possibilities for new gnosiological approaches (see ch.5, 6). Also, this indicates that Noether's theorem is a meta-theorem, having in mind that a proof obtained by a metatheory could be called a *metatheorem* (see ch.7).

4. Greek principle of causality revealed in the quantum and relativity theories

This was sometimes a paradox, but now the time gives it proof.
Shakespeare, Hamlet, Act 3, Scene 1

§ 4.1. INTRODUCTION

The concept of causality is so deeply rooted into modern thought, above all into modern physical thinking, that it is never questioned, taking the place more fundamental even than laws of conservation. It is usually taken as granted that there is only one possible principle of causality, that defined by Galileo (see PREFACE and §§ 4.2, 4.4), even when some theories of new age (Western) physics, are abandoning this principle as in the case of Theory of Relativity and Quantum Mechanics (see § 4.4.). Yet, the Ancient Greeks have developed another possible causality principle (see § 4.2), which have been put aside by modern thought, together with the results of Greek physics.

Galileo-Newtonian physics was successful in freeing itself from what they called Aristotelian physics, which was claimed to be notorious for its connection with medieval thought, but it seems that it was not so easy to free contemporary physics from some Greek points of view. Especially when, after producing new experimental evidence, the need raised for new concepts like Planck's quantum of action and hypothesis like that of Fitz-Gerald on contraction of space and dilatation of time along the light path (which was stated in order to explain the negative result of Michelson-Morley experiment), which produced even more radical result like the Lorentz transformations, and led Einstein to conclude that the Fitz-Gerald hypothesis and Lorentz transformations are not mere mathematical tricks, but logically founded truths.

So, here we are discussing in § 4.4 the relation between Galilean and Greek principles of causality and contemporary physics, after we have established in § 4.2 the concepts of causality and space as Greeks saw it and then introduced in § 4.3 the concept of time as Modern age put it. Because, as we argue in § 4.3 Greeks had not the need for defining *time*, their only interest was *existence*. Thus only Modern age has included time into its line of thought, without ever defining it, leaving this problem to future thinkers (though Bergson tried to introduce some kind of definition of time, but it was never widely accepted).

§ 4.2. **SPACE IS FUNDAMENTAL FOR GEOMETRY**

Pythagoras was the founder of Greek mathematics, because he was first to realize antic number as the principle of world order of physical things. His basic statment, expressed in a modern language, is: *A number is a substance of all things, and organization of Universe, generaly speaking, is a harmonic system of* *numbers and their relations*. There, following the Eleatic tradition and partly forgetting Heraclites views, originated the Platonic world of forms (or ideas), which are unchangable. In modern physics, motion means a continuous change in the location of a body, but for Aristotle motion ment also growing of plants and animals, and humans, too.

Aristotle wrote in his *Metaphysics* and elsewhere: “All causes of things are beginnings; that we have scientific knowledge when we know the cause; that to know a thing’s existence is to know the reason why it is.” Greek world consists, as we see from this quotation and must repeat, of objects which are appearing and disappearing, but never changing, and therefore any interaction between them is impossible. Their relation could be only non-qualitative. But it still can be numbered. As then numbers are involved, it could be concluded (wrongly) that both "procedures" of appearing and disappearing of objects, and their relations, can be modeled (explained) by arithmetical forms. Yet, for Greek thought arithmetic can only countdown, while numbers themselves cannot express any interaction, relation, so Plato and Aristotle, following Pitagoras "geometrize" numbers, developing them from geometric relations. Thus, beginning with Pitagoras in VI century BC until the end of its development, Greek mathematics was geometry.

In the world of simultaneous objects one cannot talk, using modern terminology, about the "behavior of systems", but about the "structure of systems", so the static geometric rules are perfect for such models. Yet, Greek world was full of places which never were connected together into the space in a modern (Western) manner. Indeed Galileo needed the modern concept of space, as he needed the modern concept of causality ("Every **change** of the *state* of the *object* has its cause", not "Every **object** has its cause", see Preface and § 4.4) for defining ***inertia*** of heavenly and terrestrial objects.

Thus, Modern thought could not accept Greek concepts. It needed, so to speak, some "diachrony" in all that harmony and the time as the measure of change was introduced already when the concept of functional dependence was discovered. Yet when speaking of Greek concepts, we are having in mind Eleatic tradition, which in its extremes was criticized by both Plato and Aristotle, but as it usually happens, also was incorporated into the basics of their thoughts (in defining a motion, and a so on, see the end of the first passage of current paragraph).

§4.3. INTRODUCING FUNCTIONS INTO MATHEMATICS IS REALLY INTRODUCING TIME

Newton still tried to introduce "movement" (which needs time) into the geometry by using the method of "fluxions". For the concept of inertia (which is related to the concept of "action at distance"), as already mentioned, the concept of space is *conditio sine qua non*. Thus, one cannot treat these problems without the geometry, but Greek geometry is not satisfying, because it does not include the "change" of the state of objects. Therefore Newton, sticking to Galileo's new principle of causality, tried to adjust classic (Greek) geometry to the new demands, introducing velocity into elements of geometry, making thus geometric objects subject to change with regard to their own movement. There were revealed the shortcomings of Greek concept of existence instead of time which is needed for describing movement of terrestrial and heavenly bodies.

But the world with time in it was introduced into mathematics, and for that matter into physics, also, by introducing functions. Interestingly enough, though Descartes was the first one to use functional dependence, the word function was coined by Leibniz. Only by using functions it is possible to express mathematically change, because as absolutely static system geometry that cannot achieve. And that is exactly the reason why Leibniz's calculus was more successful than Newton's method of fluxions.

The temporal properties of function can be easily seen in the next definition [7]: *A **function** is a binary relation f, with the property that for an element x (the argument of the function) there is no more than one element y such that x is related to y. This uniquely determined element y is denoted by f(x) and is called the value of the function*. This formulation includes into itself automatically the notion of succession, i.e. arguments and values of a function are not simultaneous, but successive. Thus time is obviously included when using functions.

Also, it is important to mention that the later concept of *operator* is, through its definition as a mapping of one linear vector space to the other, only an extension of the concept of function, and everything that is said for function as being "time based", or "time inclusive", can be applied to the operator, but with the distinction that operator has eigenvalues, which is important for our further discussion.

So, finally, the Newton’s aim, the wedding of geometry and motion was achieved in *analytic geometry* of Descartes, and in tensor calculus which is deeply related with the General theory of relativity.

§ 4.4. IS THE WESTERN CONCEPT OF CAUSALITY THE ONLY ONE THAT IS USED IN CONTEMPORARY PHYSICS?

It can bee seen that, accordingly, (when introducing the concept of inertia), Galileo-Newtonian physics has incorporated into itself the Western (Galilean) concept of causality, i. e. it works with *causes of the change*, and not with *causes of objects* (see PREFACE). (The difference between two is illustrated in Preface on the example of getting vapor from water, it also can be seen in the case of growing, because roughly speaking for the Greek point of view a puppy disappears and a grown dog appears.). This new cocept of causality worked also for Maxwellian Electrodynamics, and every other branch of new age physics, until the introduction of subjectivity into XX Century physics.

Exactly at the beginning of XX Century, mankind, after turning to subjectivity in Kant's philosophy, and more than a century later to subjectivity in the modern novel in works of "The Saint Trinity" of modern literature: Proust, Joyce and Kafka[3], introduced subjective point of view into contemporary science, or to be precise, into contemporary physics, through works of Einstein, Bohr, Heisenberg, Schrödinger, von Neumann, DeBroglie and Dirac, to mention the few most prominent ones. This turning to the "point of view of an observer" really began with Einstein's *Special theory of relativity*, and his applying Planck's idea of *quantum of action* to the problem of photo effect in 1905, but we shall focus here on his later work of which first ideas were published in 1907. In paper [10] Einstein introduced the thought experiment with an elevator which is falling freely in gravitational field [Einstein himself likes to call that object a *chest*, see paper 10, also 9], and is carrying the time and space with itself, which is in fact the Greek concept of space as a *place*, although including time, for introducing which there was not, as mentioned earlier, need in Greek thought, and putting it on the same basis as space.

Also, as mentioned in the Introduction, Bohr Postulates include into themselves the notions of ***appearing*** and ***disappearing*** of objects. And this is not only the case in Old Quantum Theory, but in Matrix Mechanics of Heisenberg, too. Namely, as already explained, Bohr established the foundation for laying down the Greek principle of causality underneath all formulations of Quantum Mechanics (never mentioning it, and probably never knowing it). Accordingly in

[3] Symbolically speaking, as Kafka's works were published after the great events in physics we are presenting here. So, paradoxically, there is some simultaneity in all this miraculous development of human thought.

Schrödinger's *wave mechanics*, the Greek principle of causality was accepted through the eigenvalues of operators (introduced into Quantum theory by Born and Wigner), representing physical quantities. Eigenvalues which represent the orbits of electrons that are changed by *appearing* and *disappearing* of electrons. Then there was needed, as Messiah [22] put it, a voyage to Copenhagen: "Copenhagen interpretation" which deepens Heisenberg's putting together the two principles of causality via *Uncertainty Relations* (as already stressed, without explicitly noticing it). The Copenhagen interpretation of Quantum Mechanics tries to make this connection by introducing *probability*, thus avoiding making choice between two concepts. This is, probably, why the "Copenhagen interpretation" was so unacceptable for Einstein intuitively (he, of course, tried to realize his intuition introducing EPR-paradox, etc.). But, as is already mentioned, Einstein's tries failed in a sense that Quantum Mechanics survived. And this survival was so overwhelming that we now have in Relativistic Quantum Mechanics, or in Quantum Field Theory, lots of operators of creation and annihilation, charge operators which create and annihilate particles and so on.

Now if one goes back to the beginning of application of II Noether's theorem, when infinite continuous transformations of field functions and coordinates are applied to relativity theory to show that it does not violate the law of conservation of energy there are two possibilities of using group theoretical interpretations – passive and active [22]. Yet passive interpretation, as it is a rotation of coordinate system, is basically related to the Western principle of causality, while active interpretation is deeply connected to the Greek principle. Thus Noether's theorem, which introduces group theoretical aspects into defining the law of conservation of energy in the theory of relativity, explains to us where the Greek principle of causality is sneaking into relativistic physics.

Though introducing the Greek principle of causality into physical theories started as early as the beginning of XX Century, it was not recognized until now [11]. So we here tried to show that this concept is built in the body of most of the contemporary physics, but first we had to underline the difference between the Western (Galilean) and Greek principles of causality, and to shed some light, from our point of view, on the problem of differences in concepts of space and time between Western and Greek thought. That discussion, aside from giving the new insight into problems of causality, space and time, gave us opportunity to stress the importance and potentials of Noether's theorem.

5. New gnosiological aspects of Noether's theorem

Be thou as pure as snow, as chaste as ice…
Shakespeare, Hamlet, Act 3, Scene 1

§ 5.1. Introduction

It is a well known fact that theories describing behavior of atoms in strong laser fields, treat an atom as quantized object and electromagnetic field classically. Yet these theories which combine classic and quantum approach have shown their vitality not only in the case of strong laser fields but also for super strong fields [see, for instance, 11, for so called ADK-theory], and even when relativistic effects are included. Let such theories be called ″mixed″. For more details about ADK-theory as a »mixed« theory see, for instance, [12-14].

In order to check the reliability of *mixed* theories, concerning conservation laws, here shall be used Noether's theorem. That means that a rather new gnosiological concept is proposed, or pretty renewed one. Because, Noether's theorem is not applied to obtain laws of conservation of natural, physical objects and concepts, which is to the general opinion, the main and most important purpose of this theorem, but to a somewhat different goal. I.e. to check the validity of some theories from the point of view of their fitting to describe natural processes which are obeying (as all natural processes are, in fact) the laws of conservation of physical quantities such as energy, momentum, angular momentum etc. As it was mentioned in Introduction, interestingly enough, Noether's theorem was even discovered to check one theory (General Theory of Relativity) in that sense, but this has been ever since forgotten, and this theorem was recognized as the most useful tool for connecting symmetries and conservation in modern physics.

Maybe it was too early then, in the beginning of XX century, to use the concept of the metatheory in scientific research. This concept was developed by Hilbert at that time and was not so widespread. Hilbert's concept was related to the same concept introduced in General Liguistics witch Ferdinand de Saussure layed out in his outstanding book of the same name [22]. One can notice that General Linguistics tries to take the same role among Social Sciences and theory of literature, as mathematics has among Sciences, which is in fact the metatheoretical role. Emmy Noether, who was working with Hilbert, among others, was the most suitable person to move forward with such ideas and to even discover something that can be considered a metatheorem.

§ 5.2. Corollary to Noether's theorem

Following ch. III, we could repeat the next formulation of Noether's theorem: To any *s*-parametric continuous transformation of field functions and coordinates, which keeps variation of action zero, there correspond s-dynamic invariants (i.e. constant in time combinations of field functions and their derivatives).

In what follows the results and notation of ch.3 shall be used, the Greek indices are denoting 4 space-time coordinates (0,1,2,3) and Latin are denoting spatial 3-coordinates (1,2,3). So, one should start with expression (3.7) from ch.3, adjusted to our present needs

$$\Theta_{\nu}^{\kappa} = -\frac{\partial L}{\partial u_{\alpha;\kappa}}(u_{\alpha;\lambda}X_{\nu}^{\lambda} - \Psi_{\alpha\nu}) - L(x)X_{\nu}^{\kappa}, \quad (5.1)$$

which appears in the action integral obtained for s-parametric transformations (here L(x) is the Lagrangian of the system, and covariant derivation is denoted by ";"). Standard procedure is to introduce infinitesimal transformation of 4-coordinates, for which infinitesimal space-time rotations could be chosen:

$$x^{\kappa} \rightarrow x'^{\kappa} = x^{\kappa} + \delta x^{\kappa} \quad (5.2)$$

and, for transformation of field functions, one has

$$u_{\alpha}(x) \rightarrow u'_{\alpha}(x') = u_{\alpha}(x) + \delta u_{\alpha}(x). \quad (5.3)$$

Variations δx^{κ} and δu_{α} could be expressed using infinitesimal linearly independent parameters of transformation $\delta\omega^{\nu}$:

$$\delta x^{\kappa} = \sum_{1\le\nu\le s} X_{\nu}^{\kappa}\,\delta\omega^{\nu}, \quad \delta u_{\alpha}(x) = \sum_{1\le\nu\le s} \Psi_{\alpha\nu}\,\delta\omega^{\nu}, \quad (5.4)$$

where s is the number of parameters of transformation and is not restricted by our confining Latin indices to three dimensions, i.e. it could be a number greater than three; $\delta\omega^{\nu}$ are parameters themselves.

So, if one goes back to expression (5.2), choosing for parameters of transformation values δx^{κ}, one obtains, from (5.4)

$$X_{\lambda}^{\kappa} = \delta_{\lambda}^{\kappa}, \quad \Psi_{\alpha\lambda} = 0 . \qquad (5.5)$$

Because of this, Θ_{ν}^{κ} from equation (5.1) becomes a mixed second rank tensor

$$T_{\nu}^{\kappa} = \frac{\partial L}{\partial u_{\alpha;\kappa}(x)} \frac{\partial u_{\alpha}}{\partial x^{\nu}} - L\delta_{\nu}^{\kappa} , \qquad (5.6)$$

which could be transformed into fully contravariant form

$$T^{\lambda\kappa} = \frac{\partial L}{\partial u_{\alpha;\kappa}(x)} \frac{\partial u_{\alpha}}{\partial x^{\lambda}} - L g^{\kappa\lambda} . \qquad (5.7)$$

It is shown in Bogolibov's book that integrals over threedimensional configuration space, integrals of the type

$$C_{\nu}(x^{0}) = \int \Theta_{\nu}^{0} dx , \qquad (5.8)$$

are constant in time.

For tensor $T^{\kappa\nu}$ from (5.7), such an integral would give constant in time 4-vector

$$P^{\lambda} = \int T^{\lambda 0}\, d\vec{x}\,. \qquad (5.9)$$

Zero component of this vector is, in fact, Hamilton function of Classic mechanics, i.e. energy. As time and space, which are involved in transformation (5.2) that is responsible for the resulting conservation law, are not quantized in all physical theories, except, maybe, when the ultra relativictic effects and energies are involved, we can smoothly connect the part of the theory which uses classic approach with that using quantum. Thus, one has the law of conservation of momentum-energy, and especially of energy in the case of the theories which combine classic and quantum approach (we call these *mixed* theories).

If one wishes to separate the energetical part from that of momentum, one could take only double zero component of tensor (5.7), i.e. T^{00}

$$E = P^{0} = \int T^{00}\, d\vec{x} = \text{const}\,, \qquad (5.10)$$

obtaining thus the pure energy conservation law, which is the result of time translation, and because the time is not quantized, it is applicable to the *mixed* theories.

In order to illustrate previous arguments further we shall add following. If we chose 4-rotations of space-time instead of translations it would result in conservation of 4-angular momentum (i.e. 3-angular momentum and spin), but this would not be applicable to mixed theories, because the rotations which are continuus in the classic theory, have to be quantized in the quantum theory, so there is no smooth connection between the classic and quantum part of the theory, hence for such theories the conservation of angular momentum and spin is not working.

Thus here has been proven the following (this in part was published in [15]).

Corollary to Noether's theorem

Physical theories wich combine ("mix") the quantum and classic approach are subject to the laws of energy and momentum conservation, but are **breaking** *the law of angular momentum - spin conservation.*

This is the shortcoming of such theories, but one that does not affect the results of ADK-theory, for instance, because that theory never operates with spin, and also because the most probable ejected electrons described by that theory have orbital quantum number $l = 0$.

This proof, being all-inclusive, holds for other mixed theories and is not restricted to theories that describe laser-atom interaction. Yet it is applyed here to such theories solely, not only because they are most familiar, but also because one feels that there is no need for further illustration of this rather general principle.

6. Alternative approach to the problem of spin and statistics

Nymph, in thy orisons be all my sins remember'd
Shakespeare, Hamlet, Act 3, Scene 1

6.1. **INTRODUCTION**

Famous Noether's theorem[4] introduces laws of conservation directly in terms of symmetry requirements in Lagrangian (***L***). One of the advantages of this theorem is that it is easily applied to quantized field theories. This is the reason it has been often used in developing new theories as well as in textbooks [6,16,17].

Yet, as it seems, the potentials of this outstanding theorem are far from being exhausted [see, also, 15]. For instance, we are proposing here the new application of Noether's theorem to the problem of the origin and conservation of spin, drawing, also, the consequences which explain the need for fermions to be described by antisymmetric functions, thus, unlike bosons, being subject to Pauli exclusion principle. Though those results are already obtained [18, 22] in the frame of Quantum Field Theory, it could be of certain interest to show their alternative obtaining in the frame of Standard Quantum Theory, because mathematics is straight forward, and the physical interpretation may be fruitful in its innovations.

[4] To any S-parametric continuous transformation of field functions and coordinates, which keeps variation of action zero, there correspond S-dynamic invariants (i.e. constant in time combinations of field functions and their derivatives) [ch.3,6].

The line of thought is the following: From infinitesimal, belonging to the continues Lie group, 4-rotations

$$\bar{x}^{\nu} = x^{\nu} + x_{\mu}\delta\Omega^{\nu\mu}, \qquad (6.1)$$

where with $\bar{x}^{\nu}$ we denote new, transformed, 4 - coordinates, the law of conservation of 4 - angular momentum follows (that is, of course, orbital angular momentum and spin are conserved). Also, from infinitesimal 3-rotations the law of conservation of orbital angular momentum should follow [and this can easily be shown, 8], as is the case, for instance, for 3-dimensional space translations, which give the conservation of momentum. Accordingly, as the time translation results in the conservation of energy, if the pure time rotation were introduced, some conservation connected with spin should follow. But of course, for rotation one needs more than one dimension. So, our time should have at least two dimensions (and maybe more)[5]. It has been said that such theories have difficulties with causality, but new development of the two-time physics has shown that in the case of two dimensional time the gauge can be found, which resolves the problem of causality and ghosts (negative norm states) [19].Yet it seems "we can assume that the number of times is greater than 2, but than one does not have enough constraints to eliminate all the possible ghosts" [20]. Still, as from our reasoning it follows that rotation

---

[5] At this point we tried to speculate and rotate the time coordinate among space coordinates. But mathematics was strict; no reasonable result could be obtained.

of time coordinate will result in explanation of the origin of Pauli principle (see end of paragraph 2), only if two time was used with three time, and so give us a deeper insight into the nature of spin, then we may be forced to reach for the hypothetical solution of the problem of ghosts. Hopping that "search for missing constrains" will give results soon, as the difficulty mentioned in [20], is only of technical nature, and not the principal one.

This difficulty could be overcome, for the moment, by introducing different time which is connected with each set of elementary particles (that is to say with fermions and with bosons). So, as we shall soon see, fermions are connected with two-time physics and are originated from that kind of manifold legally [see comment in the previous passage]. Yet, bosons, which are responsible for interactions and, as such, also fundamental to our picture of the world, are carrying with themselves the three-dimensional time which is full of ghosts so we have to treat these dimensions as the ones that are not actual in the everyday physical world, but are under all circumstances bound. As this approach could give us a deeper insight into the nature of spin, we are forced to use this seemingly fictive manifold. Yet the physical values could be divided, in respect of their treatment in Noether's theorem, into spacelike (momentum, angular momentum), and timelike (energy, spin).

The former discussion relates fermions, matter building particles, to the two-time physics, and bosons, the carriers of interaction, to the three-time physics. The two-time physics is being developed by various physicists [see 19,20, and references thereof], and could be considered already established, yet three-time physics still has problems, but obviously, if we want to describe interactions in our world we have to introduce three-dimensional time, one way or another. *Also, this indicates that hypothesis of gravitons having spin 2 cannot be taken seriously into account, because it needs more time dimensions (6!) which is obviously not easy to satisfy.*

There, also, have been attempts to obtain the spin conservation, but based on an extension of phase-space via Grassmannian variables (such an attempt is illustrated in [121]) and not on the extension of notion of time. In [17] the authors touched the problem of using "time approach", though without going through all the consequences. Besides, in nature there is no, strictly speaking, the conservation of spin as there is the conservation of angular momentum, because, for example the spin selection rules are, as one knows, only approximative.

Even so, there is a candidate for conservation of spin. Indeed, elementary particles never change from fermions to bosons, and vice versa. So, this is the conservation to be sought for via Noether's theorem (see, the end of the following paragraph). This is the fact that is being confirmed many times experimentally, and also proven using completely different thinking by Pauli [18, 22]. Also, accepting the idea of multidimensional time which every set of elementary particles carries with itself, one can easily explain the problem why only fermions are subject to Pauli exclusion principle {see discussion at the end of the following paragraph) without ever using the results and methods of Quantum Field Theory like in spin-statistics theorem of Pauli.

6.2. OBTAINING SPIN CONSERVATION VIA NOETHER'S THEOREM

Now we are turning again to eq. (3.7) from ch.3, also using Greek indices to denote 4 space-time coordinates (0,1,2,3) and Latin for spatial 3-coordinates (1,2,3), so we shall start with expression (3.7)

$$\theta^{\kappa}_{(\nu)}(x) = -\frac{\partial L}{\partial u_{\alpha;x}}(\psi_{\alpha(\nu)} - u_{\alpha;\lambda}X^{\lambda}_{(\nu)}) - L(x)X^{\kappa}_{(\nu)}, \quad (6.2)$$

which appears in the action integral obtained for S-parametric transformations (here L(x) is the Lagrangian of the system). Standard procedure is to introduce the infinitesimal 4-rotations (1), where as the parameters of transformation could be chosen six linearly independent parameters: $\delta\omega^{\nu\mu} = \delta\Omega^{\nu\mu}, \nu < \mu$.

After some calculations, and obtaining expressions

$$X^{\kappa}_{\nu\mu} = x_{\mu}\delta^{\kappa}_{\nu} - x_{\nu}\delta^{\kappa}_{\mu}, (\nu \le \mu), \quad (6.3)$$

and

$$\psi_{\alpha\nu\mu} = A^{\beta}_{\alpha\nu\mu}u_{\beta}(x), \quad (6.4)$$

where, for vector fields,

$$A^{\beta}_{\kappa\alpha\lambda} = g_{\alpha\kappa}\delta^{\beta}_{\lambda} - g_{\alpha\lambda}\delta^{\beta}_{\kappa}, \kappa \le \lambda, \quad (6.5)$$

one can get the 4-angular momentum tensor

$$M^{\kappa}_{\lambda\mu} = \frac{\partial L}{\partial(\partial u_\alpha / \partial x^k)} \left\{ \frac{\partial u_\alpha}{\partial x^\lambda} x_\mu - \frac{\partial u_\alpha}{\partial x^\mu} x_\lambda \right\} + L(x_\lambda \delta^{\kappa}_{\mu} - x_\mu \delta^{\kappa}_{\lambda}) -$$

$$\frac{\partial L}{\partial(\partial u_\alpha / \partial x^\kappa)} A^{\beta}_{\alpha\lambda\mu} u_\beta(x) = (x_\mu T^{\kappa}_{\lambda} - x_\lambda T^{\kappa}_{\mu}) - \frac{\partial L}{\partial(\partial u_\alpha / \partial x^\kappa)} A^{\beta}_{\alpha\lambda\mu} u_\beta(x).$$

(6.6)

It is easily seen that the first term in the last part of expression (6.6) corresponds to an orbital angular momentum of the wave field (see § 3.2), and the second part, which shall be denoted in the following manner

$$S^{\kappa}_{\lambda\mu} = -\frac{\partial L}{\partial(\partial u_\alpha / \partial x^\kappa)} A^{\beta}_{\alpha\lambda\mu} u_\beta(x), \qquad (6.7)$$

characterizes the polarization properties of the field, and in the quantized case corresponds to the spin of the particle described by the quantized field. Which is already standard, well established result, beyond any doubt.

But to perform decoupling of the orbital momentum and spin in the theory based on assumptions (6.3-5) is not possible. So we are suggesting going the other way round, i.e. to use methods of obtaining isotopic spin (see [6]), for standard spin.

Let us deal now with rotations in two-dimensional time continuum related to the one set of particles (fermions, as shall be revealed later on, and especially at the end of this section). Since wave functions do not depend explicitly on coordinates of this continuum, and standard coordinates x_κ do not transform under the rotations of two-dimensional time continuum, we shall start with expressions for infinitesimal transformations only for wave functions

$$\bar{u}_\alpha = u_\alpha + \delta u_\alpha, \quad \delta u_\alpha = K^{ij}_{\alpha\beta} u_\beta \delta\omega_{ij}. \qquad (6.8)$$

Here $\delta\omega_{ij}$ are, antisymmetric in indices i, j (=1, 2), infinitesimal angles of rotation of two-dimensional time continuum.

It follows that tensor (6.6) in this case does not have orbital part, so

$$S^\kappa_i = -\frac{\partial L}{\partial u_{\alpha;\kappa}} K^i_{\alpha\beta} u_\beta, \qquad (6.9)$$

it gives only spin tensor from which it follows that the rotation of two-dimensional time continuum gives conservation of the half integral spin (but conservation here must be taken strictly as conservation of the half integral spin, i.e. ***fermions always stay fermions***):

$$S_i = \int S^0_i d\boldsymbol{x} = -\int d\boldsymbol{x} \frac{\partial L}{\partial u_{\alpha;0}} K^i_{\alpha\beta} u_\beta, \qquad (6.10)$$

d$\boldsymbol{x}$ being differential of two-dimensional time continuum.

Equation (6.10) represents the quantity which has only two components (i=1, 2) and is behaving like a spinor, which indicates its connection with half integral spin.

Yet if our time-continuum were three-dimensional our spin tensor should give us when rotated the conservation of the integral spin (i.e. ***bosons always stay bosons***):

$$S_{ij} = \int S^0_{ij} d\boldsymbol{x} = -\int d\boldsymbol{x} \frac{\partial L}{\partial u_{\alpha;0}} K^{ij}_{\alpha\beta} u_\beta , \qquad (6.11)$$

d$\boldsymbol{x}$ being differential of three-dimensional time continuum.

Contracting three-dimensional components with antisymmetric tensor of the third order ε_{ijp}, we obtain components of three-dimensional (pseudo) vector of spin (that is to say, the vector describing bosons):

$$S_i = \varepsilon_{ijp} S_{jp} . \qquad (6.12)$$

And according to equations (6.10) and (6.11) in both cases (of fermions and of bosons), there is no change of the type of particles from one to another. I. e. an electron is a fermion and it can not be changed. As mentioned already this is no new result [see 6,16], but it is the first time it has been obtained in the frame of Quantum Mechanics via Noether's theorem, and confirming the idea of multi-dimensional time.

Yet there is more to it. For if one consideres going over from the left hand coordinate system to the right one in the case of two-dimensional manifold, one sees that for this operation are needed one rotation through π and one inversion of the coordinate system, i.e. the inversion of one of its axes, say x. The determinant of the rotation is definitely +1 and of the inversion is –1. These two multiplied give the determinant of the system equal to -1.

Thus in the case of the two-dimensional time manifold one has the antisymmetry of functions involved. So, the spin function is antisymmetric in the case of the half-integral spin. This is in part Pauli principle given in a broader definition [22]. So, it is obvious that by eq. (6.10) the quantity is defined, which has to be described with anticommutative operators.

Also, in exactly the same situation, but now in three-dimensional manifold, one needs a rotation through π and two inversions of the coordinate system. The determinants of three, are, mutatis mutandis, exactly the same as above, so multiplied the three of them give the determinant of the system equal to 1. Analogously to the previous case, in the case of three-dimensional time manifold one has the symmetry of functions. Accordingly, the spin function is symmetric in the case of the integral spin. And that, together with the result for the half-integral spin, gives Pauli principle, without ever using the results and methods of Quantum Field Theory like in spin-statistics theorem of Pauli. Because, without this theorem, until now it was not clear why particles with the half-integral spin were subject to the law of antisymmetricity of functions describing particles (fermions) that have that feature, thus obeying Pauli exclusion principle [22].

§ 6.3. **LET US CONCLUDE**

We have shown rather striking result, that rotation of time results in conservation of spin. Here should be stressed, once again, that conservation is to be understood as keeping the status: *fermions stay fermions*, and *bosons stay bosons*. Also, this result leads to a deeper insight into the origin of Pauli principle, explaining that two-dimensional time manifold leads to antisymmetric states, logically described by antisymmetrical spin functions, or anticommuting spin operators, and three-dimensional time manifold produces symmetrical states, described by symmetrical spin functions, as this principle states. We are thus producing spin and statistics theorem, without ever leaving the results and methods of pure Quantum Theory.

Of course there are lots of problems open yet, and the most obvious one is the interpretation of fictive multi - dimensional time continuum. But, the extra dimensions in time continuum could be interpreted as bound by the spatial selection rules or energy requirements, and not effective in everyday physical world. It is not an entirely new situation in physics: quark confinement is the example that there are physical entities which could not be measured and yet are underlying very reliable physical theory. Also strings are not observable at the moment, but are revealing many of the, until now, poorly understood features of elementary particles, black holes etc.

This interpretation is strongly supported by the mathematics of the problem which is definitely giving the aforementioned results. The problem of spin has been extensively treated in many papers ([21, 23 - 25] to mention a few), but, as to our knowledge, never on the basis of time extension. For there is a prejudice that spin is exclusively spatial phenomenon, based on the model of spin as intrinsic rotation of a particle, which, of course, does not have a classic analogue, so the "intrinsic rotation" is a mere image.

7.Final remarks

Out, out brief candle
Shakespeare, Macbeth Act V, Scene 5

Two theorems were presented at the July 16, 1918 meeting of the Königsche Gesellschaft der Wissenschaften zu Götingen in Emmy Noether's paper *Invariante Varlationsprobleme* [2], read by Felix Klein, probably, as Emmy Noether herself was not the member of the Society., and now are known as Noether's theorem (singular form is applyied because her I theorem gained much more popularity in its applications in physics).

Yet this is an opportunity to stress how deeply can penetrate into physical and phylosophycal research, mathematical thoughts with well founded roots: So, as Noether's theorem is suited to various fields of physics, it is open to understanding the foundations of our civilisation and connecting the two principles of causality: Greek and Western (see, ch.4), to hermeneutical applications to new theory types (ch. 5), and, finally, to new interpretations of already accepted results (ch. 6). As mentioned before (ch. 1), the symmetry group of the general relativity theory is a Lie group with a continuously infinite number of independent infinitesimal generators, while the symmetry group of special relativity is the Poincare group, a Lie subgroup of the group of general coordinate transformations, which has a finite number (7) of independent infinitesimal generators. This is, let us stress again, what makes the difference between Noether's theorem I and II: the theorem I describes the case when there is a finite continuous group of symmetries, and the theorem II deals with cases covering an infinite continuous group of symmetries. And yet another feature of contemporary group theory, namely active and passive interpretation, gives possibility to include both priciples of causality: Greek and Western, into the interpretation of both theories of relativity via Noether's theorem (see ch. 4).

Thus it can clearly be seen how many consequences and applications of Noether's theorem there are. Of course this subjective insight into the problem gives just limited number of consequences and applications, and yet they are so reach in their varity that it can take away breath of any mathematical, or, for that matter, physical, mind.

Also, even such limited insight indicates that Noether's theorem is a *metatheorem*, having in mind that a proof obtained by a metatheory could be called so. Also, remembaring that the idea of metatheories in mathematics emerged in the beginning of XX century in the works of Hilbert and was related to the same concept introduced in General Liguistics witch Ferdinand de Saussure layed out in his outstanding book of the same name [26], one can notice that General Linguistics tries to take the same role among Social Sciences and theory of literature, as mathematics has among Sciences, which is in fact the metetheoretical role, so this is not only special tool of mathematical thinkig discovered by Hilbert, but is the part of more general rule of human understanding the surrounding world.

REFERENCES

[1] www.history.mcs,st-andrews.ac.uk,

[2] Nina Byers, *E. Noether's Discovery of the Deep Connection Between Symmetries and Conservation Laws,* ISRAEL MATHEMATICAL CONFERENCE PROCEEDINGS Vol. 12, (1999),

[3] M. Jammer, *The Conceptual Development of Quantum Mechanics*, McGraw-Hill Book Company, New York (1967) (Russian translation, "Nauka", Moscow 1985),

[4] Landau, Lifshiz, *Mechanics*, Nauka, Moscow (1973) (in Russian),

[5] www.mathpages.com,

[6] N. N. Bogoliubov, and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, New York, Interscience Publishers, Inc., (1959). Also see the third edition in Russian, Moscow, (1976),

[7] en.wikipedia.org

[8] V. I. Arnold.: Mathematical Methods of Classic Mechanics, “Nauka”, Moscow, 1974 (in Russian),

[9] A. Einstein, *Relativity, the Special and General Theory*, New York, Crown Publishers, Inc., (1961),

[10] A. Einstein, ”About the Principle of Relativity and its Consequences”, Jahrb. d. Radioakt. u. Elektronik, Bd. **4**, S. 411-462 (1907) (in German),

[11] J.Baudon, V.D. Bocvarski,

[12] V.M. Ristić, M.M. Radulović, and T.S. Premović, Laser Phys. Lett. **2**, No.6, 314 (2005),

[13] V.M. Ristić, M.M. Radulović, and V.P. Krainov, Laser Phys., **4**, 928 (1998); V.M. Ristić, J.M. Stevanović, and M.M. Radulović, Laser Phys. Lett. **3**, No.6, 298 (2006); N.B. Delone and V.P. Krainov: *Multiphoton Processes in Atoms*, Springer, Berlin, (2000),

[14] V.M. Ristić, J.M. Stevanović, Laser Phys. Lett. **4**, No.5, 354 (2007),

[15] V.M. Ristić, M.M. Radulović, Laser Phys. Lett. **1**, No.2, 79 (2004),

[16] Bjorkin, J. D., and Drell, S.D., "Relativistic quantum Fields", New York, Mc-Graw-Hill Book Company, (1964),

[17] Itzykson, C., and Zuber, J. P., "Quantum Field Theory", New York, McGraw-Hill Inc., (1980),

[18] Pauli W., Fierz, M.: *Nuovo Cimento* 15, 167 (1938),

[19] Itzhak Bars, Class.Quant. Grav. **18** , 3113-3130, (2001),

[20] Villanueva, V.M., Neto, J.A., Ruiz, L. and Silvas, J., J.Phys. A, **38**, 7183-7196, (2005),

[21] Berezin, F.A. and Marinov, M.S., Ann. Physics, **104**, 336-362, (1977),

[22] Messiah, A., Quantum Mechanics, II, Amsterdam, North Holland Publishing Co., (1976),

[23] Barut, A.O. and Zanghi, N., Phys. Rev. Lett., **52**, 2009-2012, (1984),

[24] Shima, K., Phys. Lett. **B 276**, 462-464, (1992),

[25] Pavsic, M., Recami, E., Rodriges (Jr), W.A., Macarone, G.D., Raciti, F. and Salesi, G., Phys. Lett. **B 318**, 481-488, (1993).

[26] Ferdinand de Saussure, “Cours de Linguistique Générale”, Payot, Paris, 1949 (Serbian translation, Nolit, Beograd 1977).